

Towards abstract and executable multivariate polynomials in Isabelle

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A cultural „gap“ between two communities.

▶ **Theorem proving:**

- ▶ Sound formal development of theories on top of a small trusted kernel.
- ▶ Computations reduced to logical inferences.
- ▶ Correct but inconvenient to use and painfully slow.

▶ **Computer algebra:**

- ▶ Elaboration of mathematics by paper-and-pencil or TP software.
- ▶ Separate implementation in mathematical software systems.
- ▶ Convenient to use and reasonably fast but highly untrustworthy.

How can we bridge this gap?

Our Starting Point: Polynomial Algebra

An Isabelle package in which the working mathematician can develop

- ▶ **Mathematical theories** based on an abstract view of polynomials.
 - ▶ Type-checked definitions and theorems.
 - ▶ Computer-supported/mechanically verified proofs.
- ▶ **Algorithms** based on the defined mathematical notions.
 - ▶ Executable with „reasonable“ efficiency (rapid prototyping).
 - ▶ Formal specification and computer-supported verification.

A single computer-supported formal framework for proving and computing with (multivariate) polynomials.

What is the polynomial written as $2x^3 - 5x + 7$?

- ▶ **Traditional:** the symbolic expression itself.

$$\left(\sum_{i=0}^n a_i x^i\right) \cdot \left(\sum_{j=0}^m b_j x^j\right) = \sum_{k=0}^{m+n} \left(\sum_{i \in \mathbb{N}_0, j \in \mathbb{N}_0}^{i+j=k} a_i \cdot b_j\right) \cdot x^k$$

- ▶ **Computer science:** an array `[7, 5, 0, 3]`

```
int[] mult(int[] a, int[] b)
{
    int m = a.length-1; int n = b.length-1;
    int[] c = new int[m+n+1];
    for (int i = 0; i <= m; i++)
        for (int j = 0; j <= n; j++)
            c[i+j] += a[i]*b[j];
    return c;
}
```

Two representations of a more fundamental concept.

The more fundamental concept is the modern view of polynomials.

- ▶ **Polynomial:** a function $[0 \mapsto 7, \dots, 3 \mapsto 3, 4 \mapsto \underline{0}, 5 \mapsto \underline{0}, \dots]$

Let R be a ring. A (univariate) polynomial over R is a mapping $p : \mathbb{N}_0 \rightarrow R, n \mapsto p_n$, such that $p_n = 0$ nearly everywhere, i.e., for all but finitely many values of n .

- ▶ **Elegant mathematics:**

$$\begin{aligned} \cdot : (\mathbb{N}_0 \rightarrow R) \times (\mathbb{N}_0 \rightarrow R) &\rightarrow (\mathbb{N}_0 \rightarrow R) \\ a \cdot b := k \in \mathbb{N}_0 &\mapsto \sum_{i \in \mathbb{N}_0, j \in \mathbb{N}_0}^{i+j=k} a_i \cdot b_j \end{aligned}$$

- ▶ **Polynomial ring:** $R[x]$

The set of polynomials with $(+)$ and (\cdot) ; variable x just denotes the polynomial $[0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 0, 3 \mapsto 0, \dots]$.

See e.g. [Winkler, 1996].

Multivariate Polynomials

What is a polynomial $3x^2y + 5yz$ in variables x, y, z ?

- ▶ **Polynomial:** a function $[(1, 1, 0) \mapsto 3, (0, 1, 1) \mapsto 5, (0, 0, 0) \mapsto 0, \dots]$

An n -variate polynomial over the ring R is a mapping $p : \mathbb{N}_0^n \rightarrow R, (i_1, \dots, i_n) \mapsto p_{i_1, \dots, i_n}$, such that $p_{i_1, \dots, i_n} = 0$ nearly everywhere.

- ▶ **Polynomial ring:** $R[x_1, \dots, x_n]$

The set of all n -variate polynomials over R ; variable x_i denotes the polynomial $[\dots, i \mapsto 1, \dots]$.

- ▶ **Isomorphism:** $R[x_1, \dots, x_n] \simeq (R[x_1, \dots, x_{n-1}])[x_n]$.

Recursive algorithms may be devised for many (not all) computational problems on multivariate polynomials.

- ▶ Polynomial division is defined on $K[x]$ where K is a field.
- ▶ But $K[x_1, \dots, x_{n-1}]$ is only a ring.
- ▶ Multivariate polynomials thus only support „pseudo-division“.

- ▶ Prune mapping:
 - ▶ Represent only exponents/monomials with non-zero coefficients.
- ▶ Univariate polynomial representations:
 - ▶ Dense: coefficient sequence $[c_0, \dots, c_n]$
 - ▶ Sparse: exponent/coeff. sequence $[(e_0, c_0), \dots, (e_r, c_r)]$ with $e_i < e_{i+1}$.
- ▶ n -variate polynomial representations:
 - ▶ Recursive: univariate polynomial whose coefficients are $(n - 1)$ -variate polynomials (represented densely or sparsely).
 - ▶ Distributive: monomial/coefficient sequence $[(m_0, c_0), \dots, (m_r, c_r)]$ (typically represented sparsely).
 - ▶ Total order on monomials required for unique representation.
- ▶ Algorithmic efficiency:
 - ▶ Recursive algorithms based on isomorphism operate most efficiently with recursive representation.
 - ▶ Buchberger's Gröbner bases algorithm processes terms in any given „admissible“ order and profits from distributive rep. in that order.

General Approach

abstract type

representations	dense	sparse
recursive	✓	✓
distributive	✓	✓

General Approach

abstract type ————— Map from monomials to coefficients

- Elegantly define basic operations
- Conveniently express algorithms, theorems, and proofs

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- Theorems are preserved
- Representation values can instantiate variables of abstract type.



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Every abstract algorithm is executable with every representation type.

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'a poly-rec

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Implementation in Isabelle

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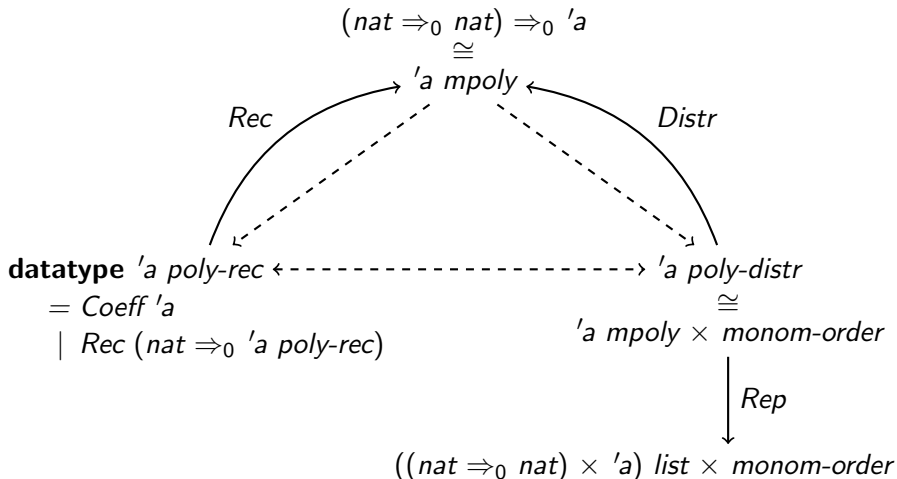
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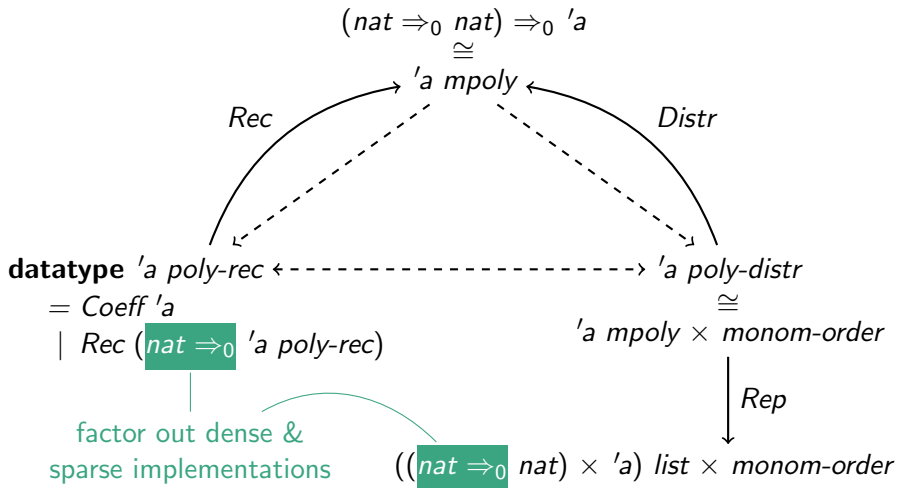
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$$('a, 4 + 3) \text{ mpoly} \neq ('a, 2 + 5) \text{ mpoly}$$

- ▶ Polynomials over an unbounded number of variables

Derived notion **variable number**:

the highest index of a variable with non-zero coefficient.

Implicitly extend polynomials as needed.

Exploit the Representation in Abstract Algorithms

Example:

- ▶ Gröbner bases algorithm depends on a **monomial order**.
- ▶ Efficiency relies on fast access to leading monomial in that order.

'a mpoly vs. *'a mpoly × monom-order*

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- ▶ Algebraic type classes require uniqueness of polynomials.
- ▶ Algorithm receives representational details as parameter.
- ▶ If polynomial's representation fits to the parameter, execution is **fast**.
Otherwise, convert polynomial ... or search ...
- ▶ No static checks, no efficiency guarantees!

Open Problem: Controlling Representations

- ▶ What happens when we combine two polynomials?

+	Rec	Distr.
	Rec	???
	???	Distr

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		Rec	???
		???	Distr

How can we make contextual information available?

- ▶ How can the user specify the representations?

value (2 :: *int poly*) * 3

Recursive or distributive? Dense or Sparse? Which monomial order?

In CAS, the user declares his choice as a configuration option.

Can we mimick this in Isabelle?

The Ubiquitous Type Class *zero*

typedef 'a \Rightarrow_0 'b = { f :: 'a \Rightarrow 'b :: *zero* | *almost-everywhere-zero* f }

There is no map function for 'b that satisfies

$$\text{map } f \circ \text{map } g = \text{map } (f \circ g)$$

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BNF \Rightarrow_0 is not a BNF!

Must construct *'a poly-rec* manually

Lifting Quotient theorem only for relations that respect *zero*

No parametrised correspondence relations

Transfer Transfer rules must restrict function space

\implies is too weak

Library Re-implement finite maps with the invariant $0 \notin \text{ran } m$

How can we improve reuse?

Current state of multivariate polynomials in Isabelle:

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- ⊕ Prototype of abstract and representation types with minimal set of operations
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Up for discussion:

- ▶ User-friendliness/convenience for the working mathematician.
- ▶ Control of representations
- ▶ Better integration with Isabelle packages