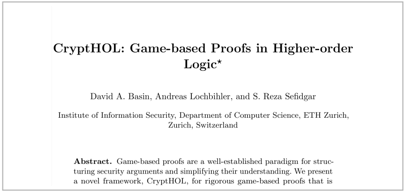
A Mechanized Proof of the Max-Flow-Min-Cut Theorem for Countable Networks

Andreas Lochbihler

Digital Asset

Motivation

CryptHOL



CryptHOL: Game-based Proofs in Higher-order Logic*

David A. Basin, Andreas Lochbihler, and S. Reza Sefidgar

Institute of Information Security, Department of Computer Science, ETH Zurich, Zurich, Switzerland

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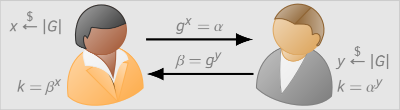
Relational logic for discrete probability distributions



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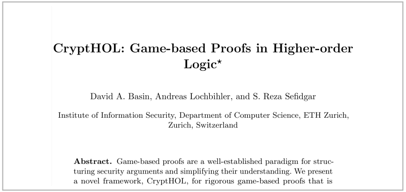

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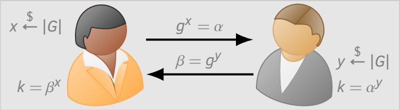
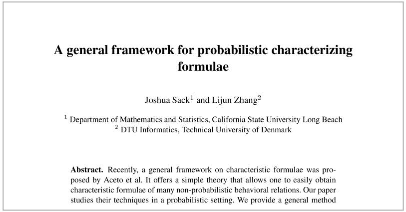


Lifting operator



A general framework for probabilistic characterizing formulae

Joshua Sack[1] and Lijun Zhang[2]

[1] Department of Mathematics and Statistics, California State University Long Beach
[2] DTU Informatics, Technical University of Denmark

Abstract. Recently, a general framework on characteristic formulae was proposed by Aceto et al. It offers a simple theory that allows one to easily obtain characteristic formulae of many non-probabilistic behavioral relations. Our paper studies their techniques in a probabilistic setting. We provide a general method

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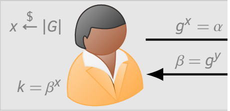
Relational logic fc probability distrib



The Max-Flow Min-Cut theorem for countable networks

Ron Aharoni[a,1,3], Eli Berger[b,2], Agelos Georgakopoulos[c,3], Amitai Perlstein[a], Philipp Sprüssel[c,3]

[a] *Department of Mathematics, Technion, Haifa, Israel 32000*
[b] *Department of Mathematics, Faculty of Science and Science Education, Haifa University, Israel 32000*
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A R T I C L E I N F O

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A B S T R A C T

We prove a strong version of the Max-Flow Min-Cut theorem for countable networks, namely that in every such network there exist a flow and a cut that are “orthogonal” to each other, in the sense that the flow saturates the cut and is zero on the reverse cut. If

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Lifting operator

A recent mathematical theorem

✓ formalized in Isabelle/HOL

✓ found and fixed many mistakes and glitches

✓ simpler variant with additional assumptions

[...]work for probabilistic characterizing formulae

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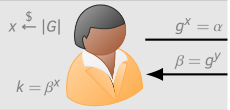

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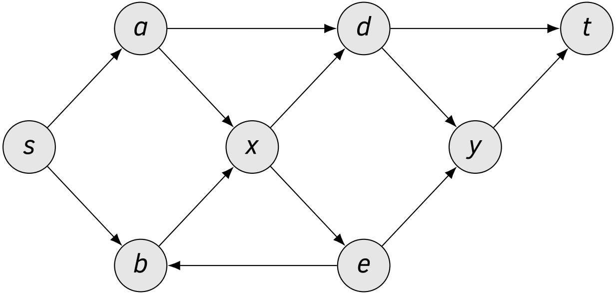
The Max-Flow-Min-Cut Theorem for Finite Networks



Network

- graph $G = (V, E)$

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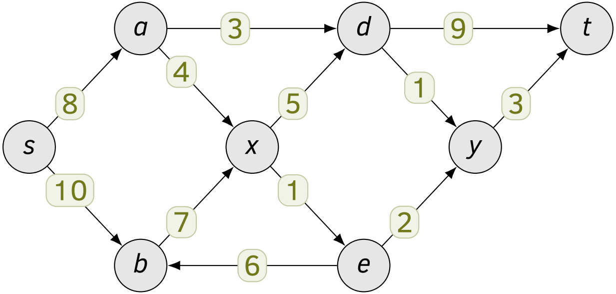

Network

- graph $G = (V, E)$
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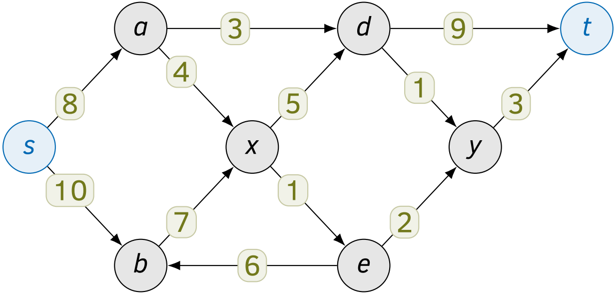

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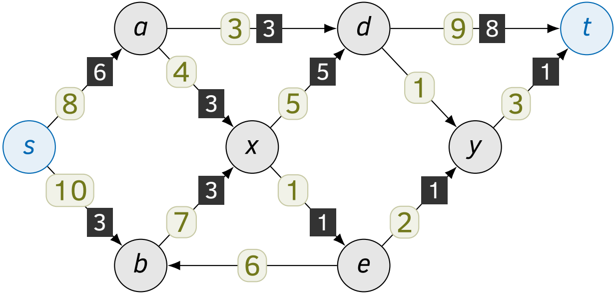

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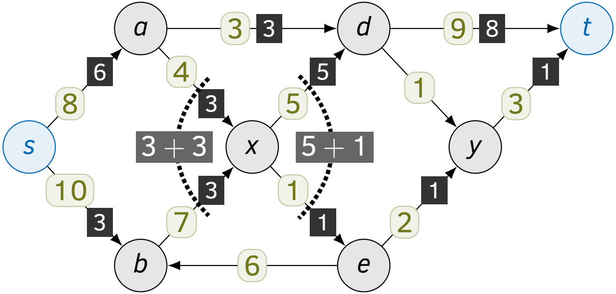

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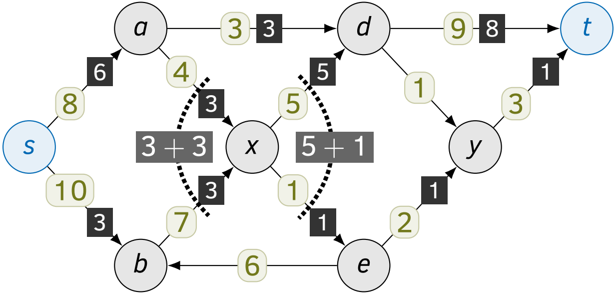

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$|f| = 6 + 3 = 8 + 1$

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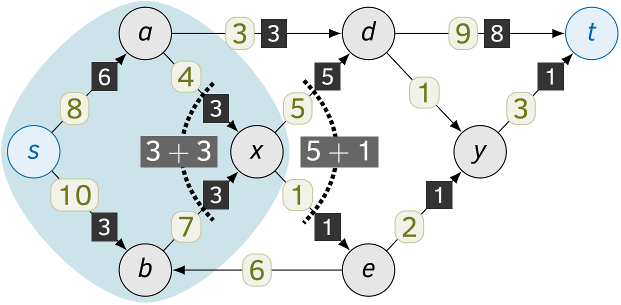

Cut $C \subseteq V$
- $s \in C$, $t \notin C$
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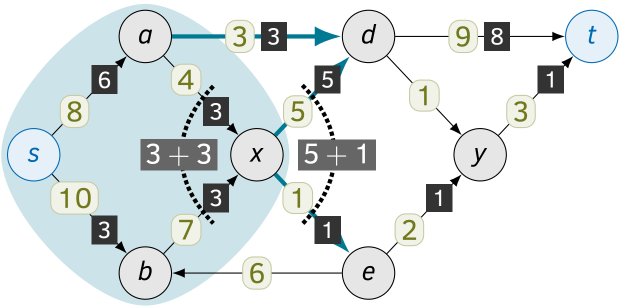

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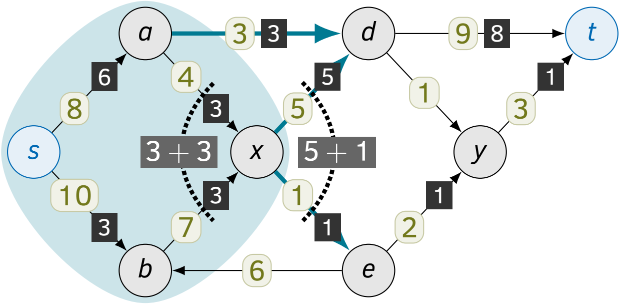

Max-Flow Min-Cut Theorem
In every finite network, there exist a cut C and a flow f s.t. $|C| = |f|$.

Lammich and Sefidgar [ITP 2016]

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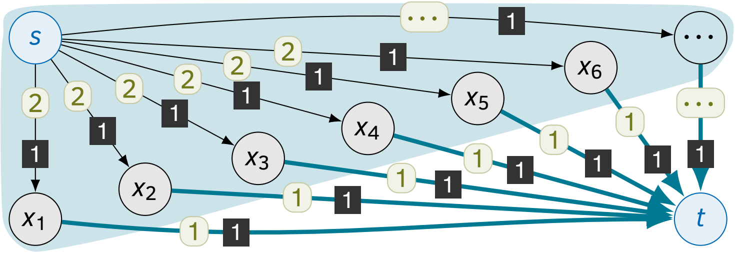
Challenges with Countable Networks



$C = \{s, x_1, x_2, \ldots\}$ $\quad |C| = \infty$

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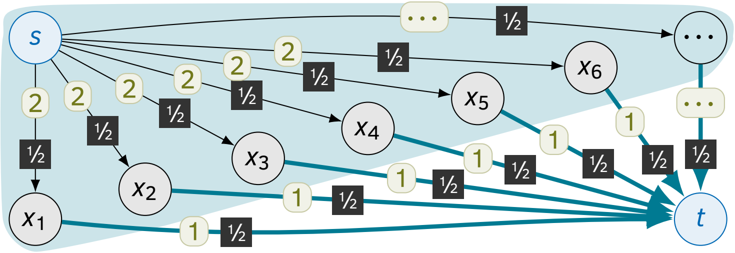
Challenges with Countable Networks



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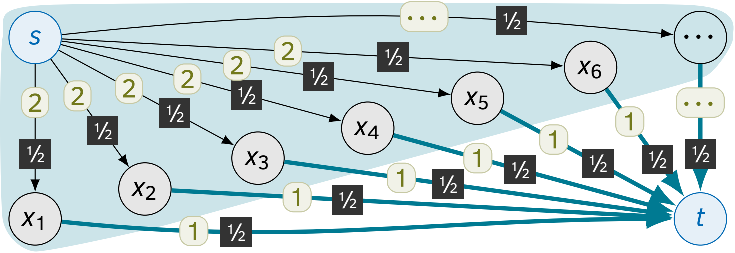

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Avoid infinite sums!

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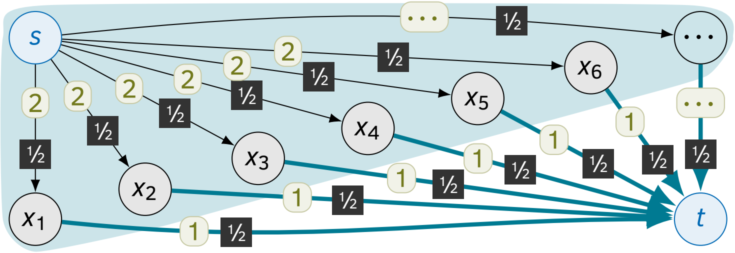

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$\cancel{g(e) = 1/2}$ $\cancel{|g| = \infty}$

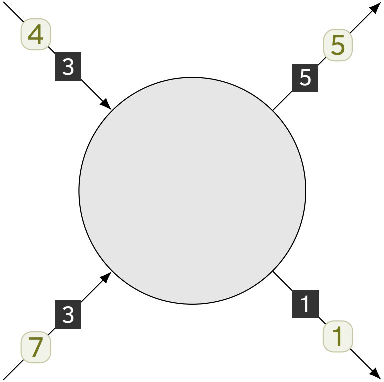
Avoid infinite sums!

Max-Flow Min-Cut Theorem [Aharoni et al.]

There exist a cut C and a flow f s.t.

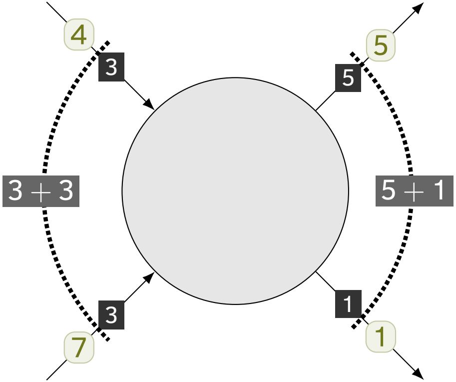
- $f(x,y) = c(x,y)$ for $(x,y) \in E, x \in C, y \notin C$
- $f(x,y) = 0$ for $(x,y) \in E, x \notin C, y \in C$

More Infinite Sums



More Infinite Sums

Flow preservation



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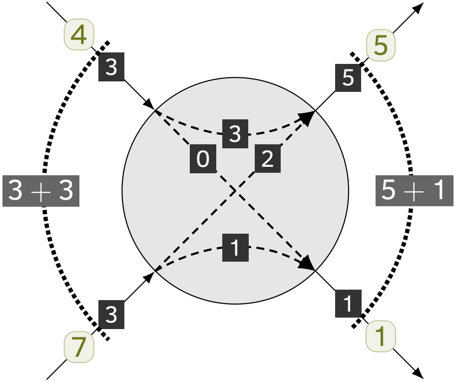

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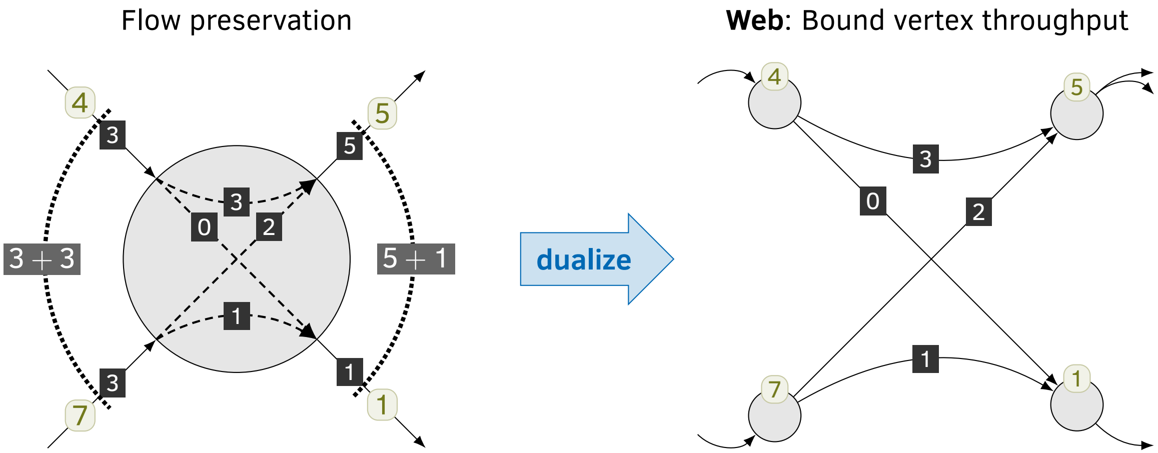

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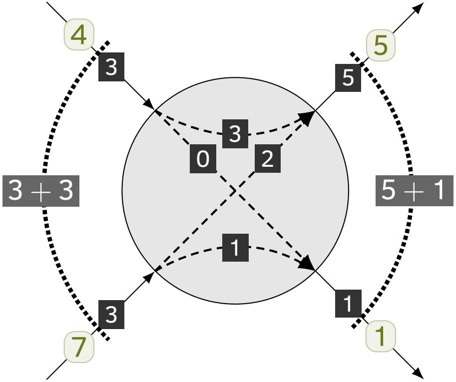
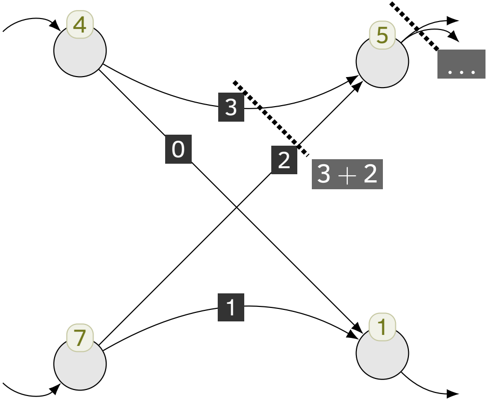


dualize

Web: Bound vertex throughput



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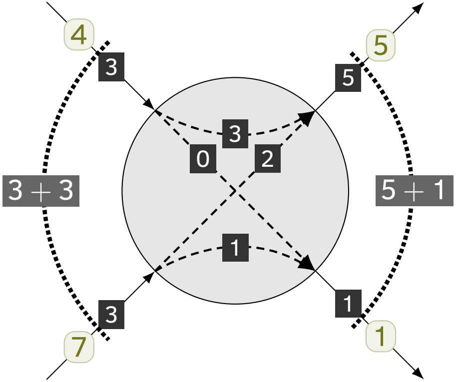

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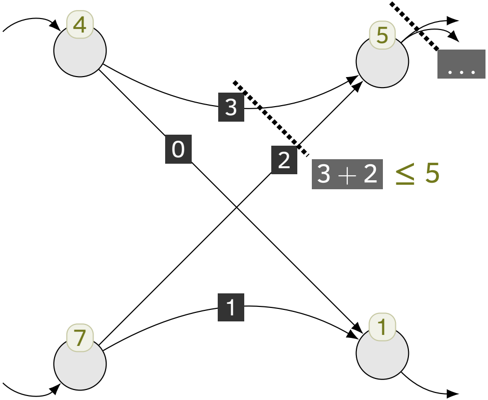

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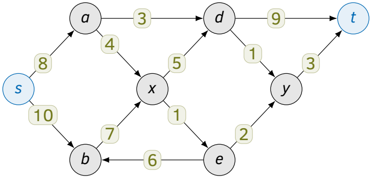

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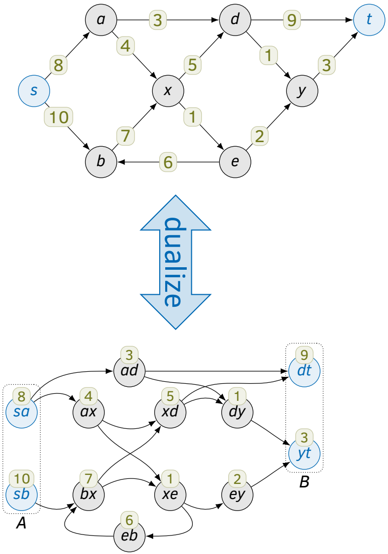

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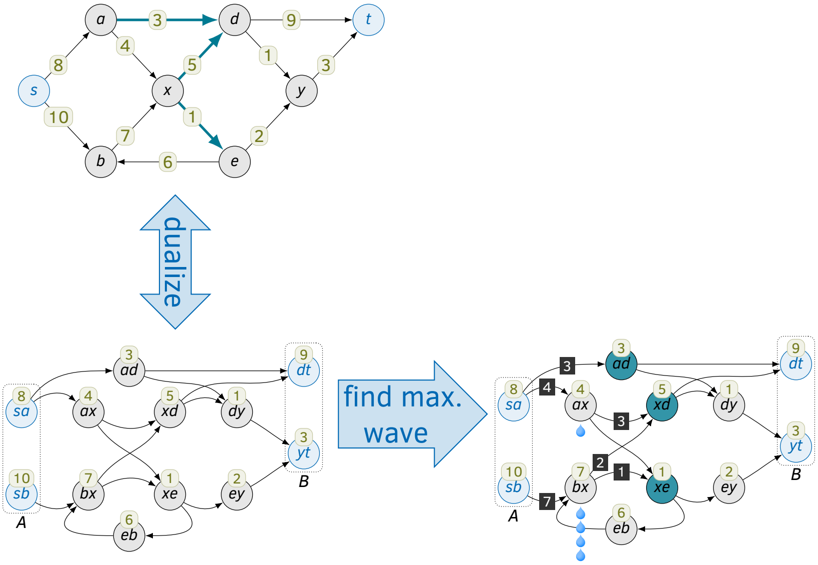

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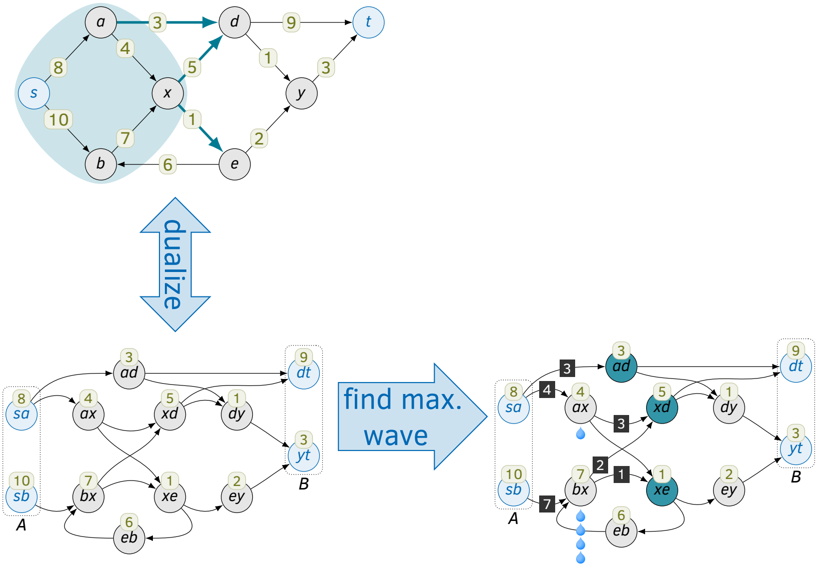

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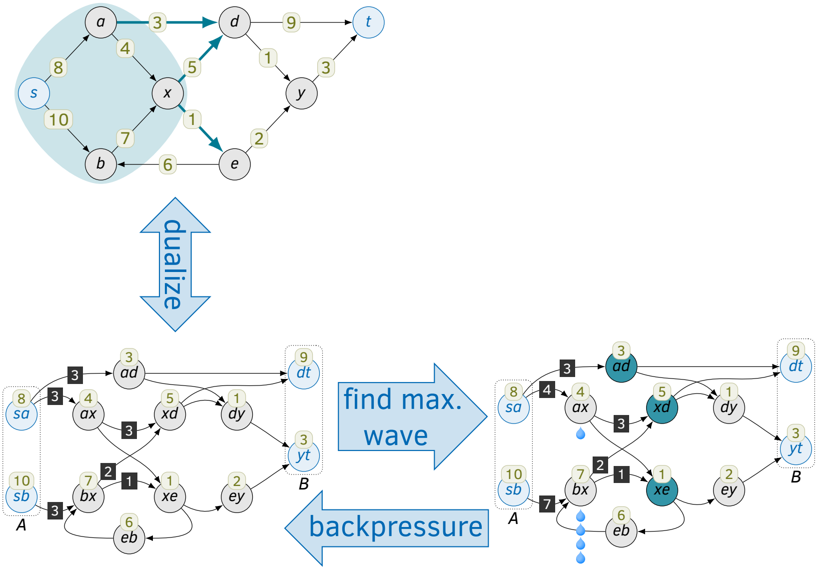

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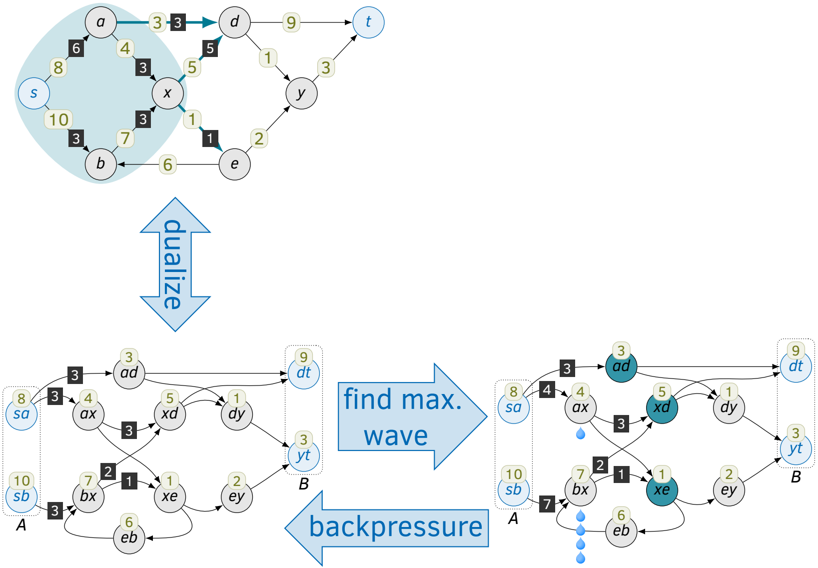

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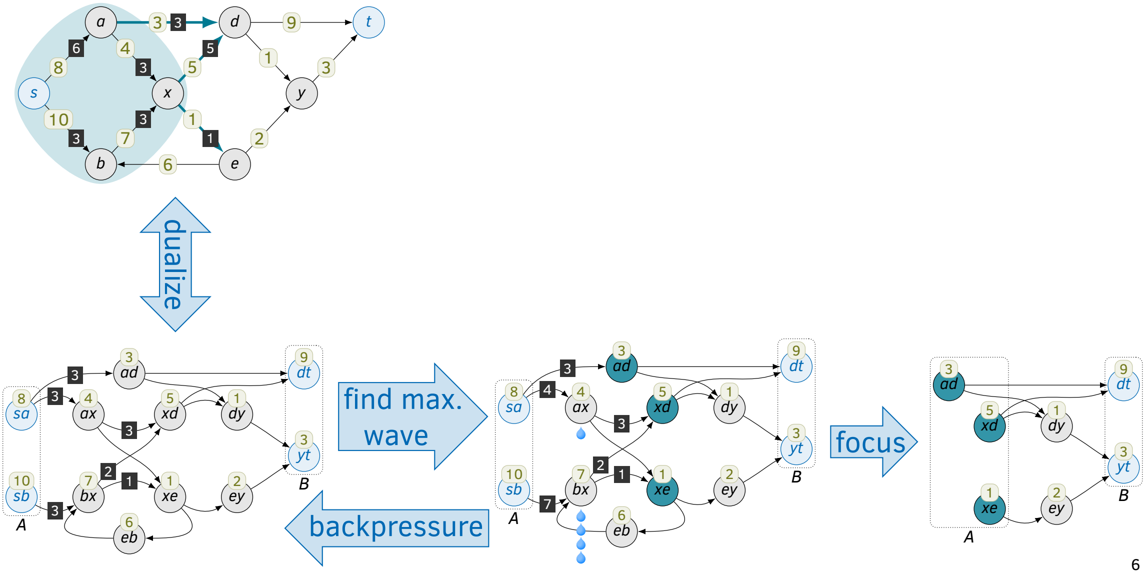

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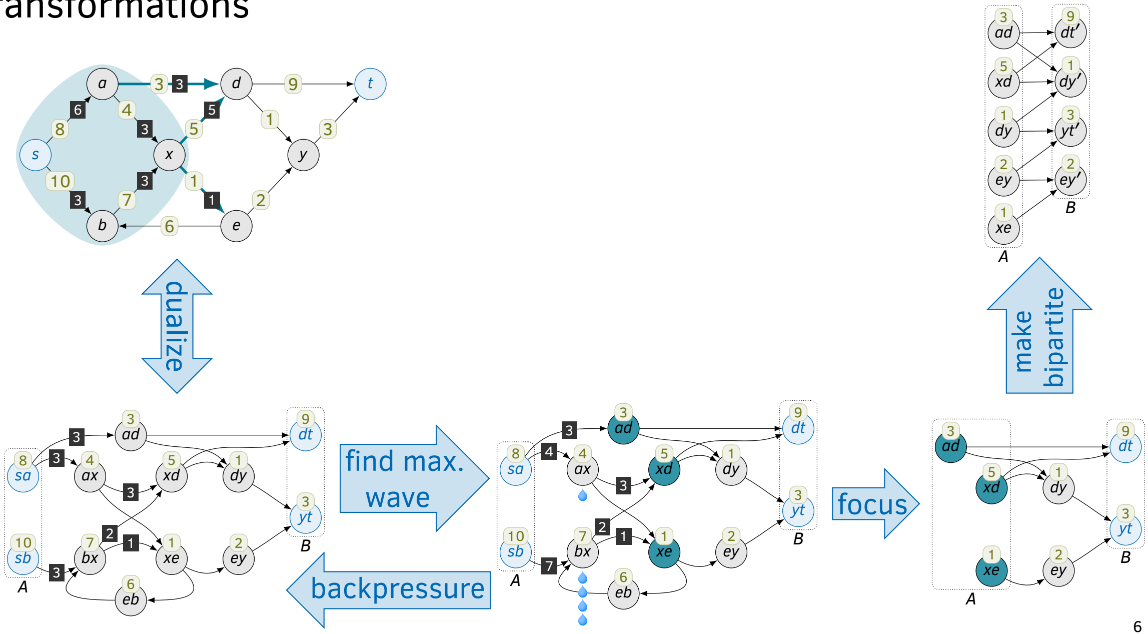

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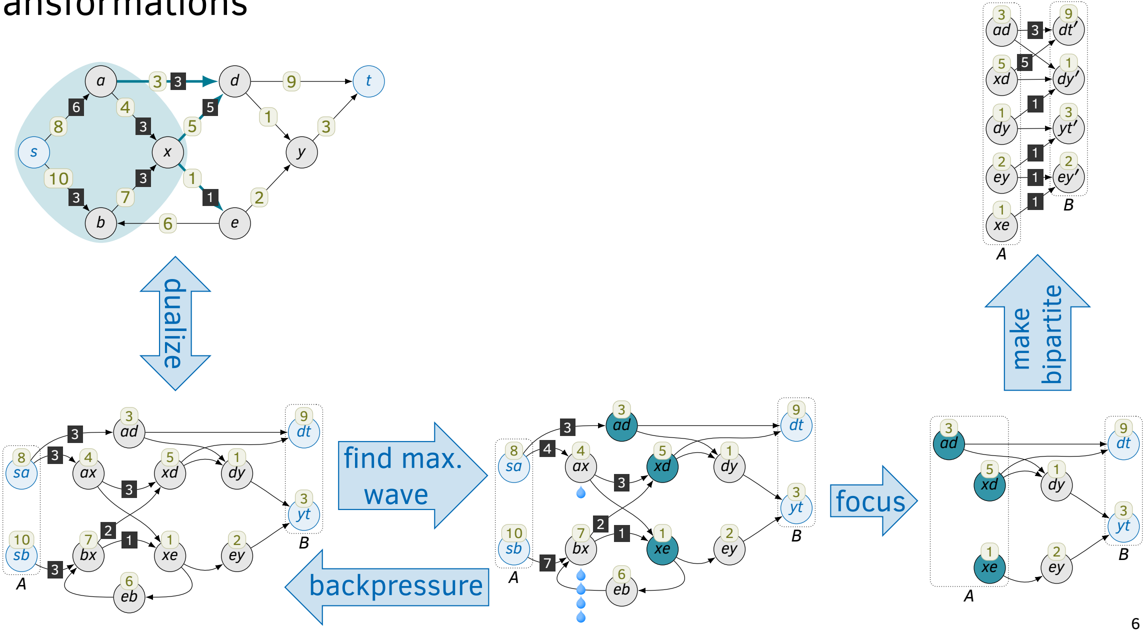

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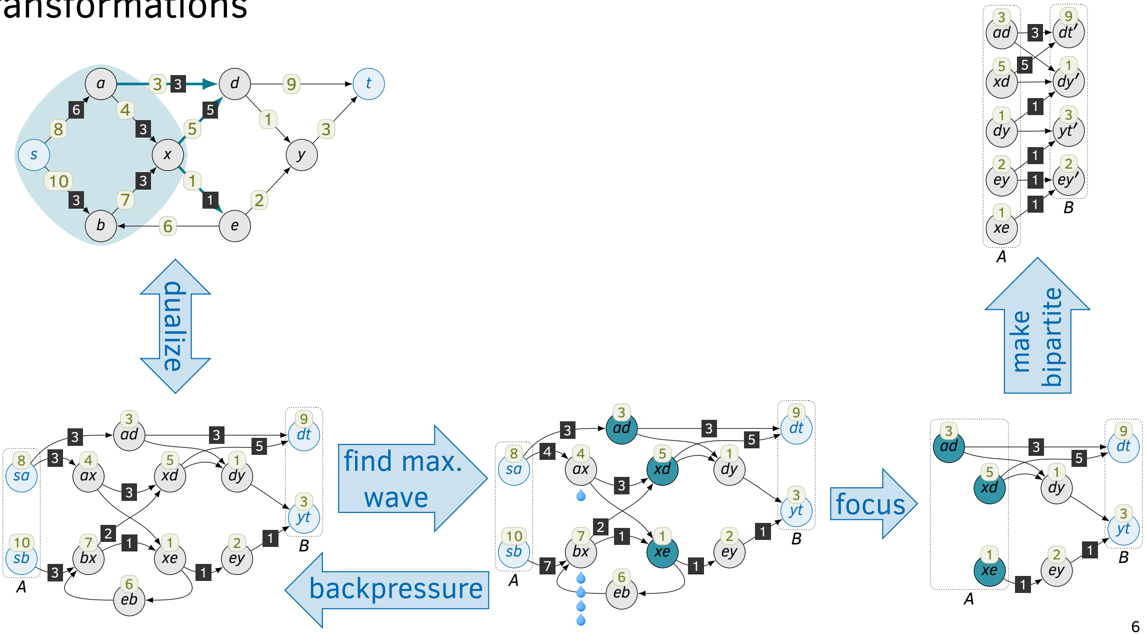

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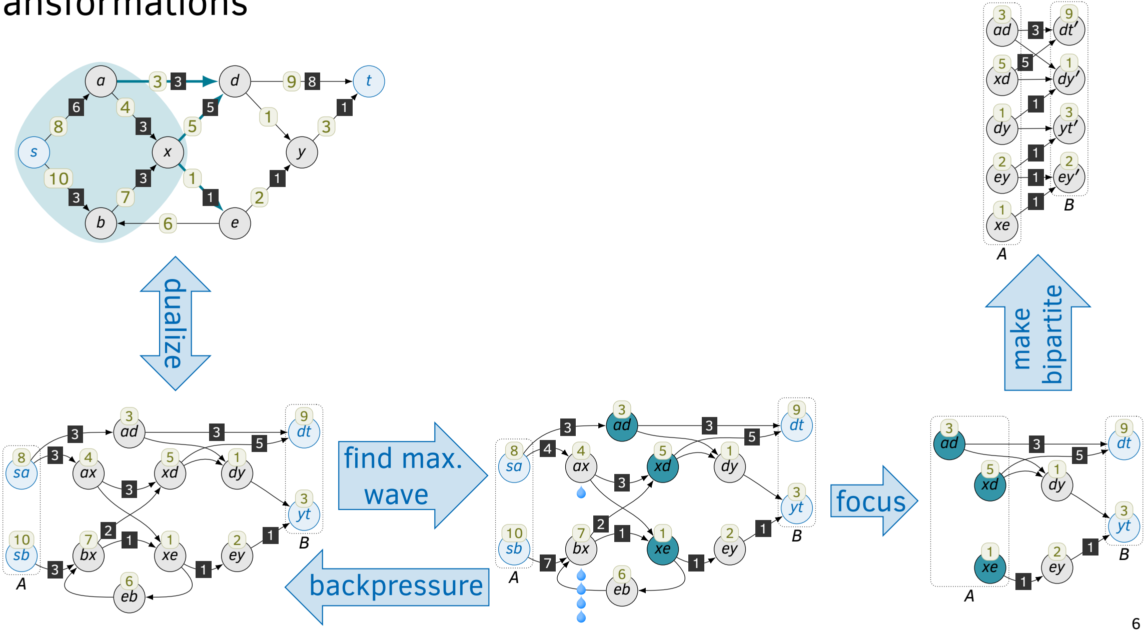

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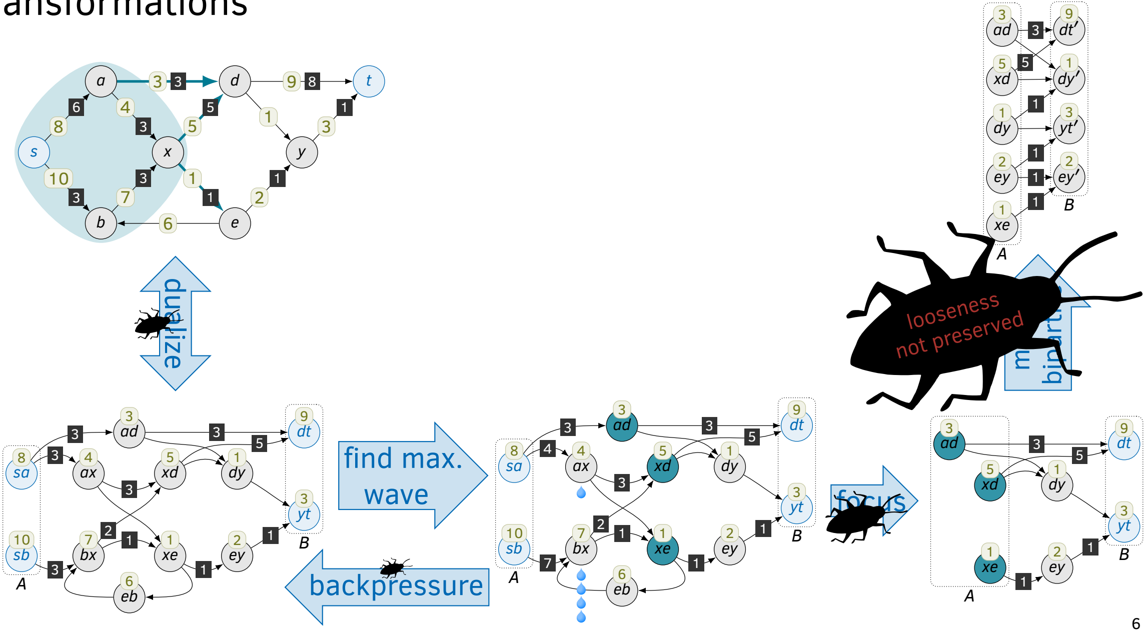

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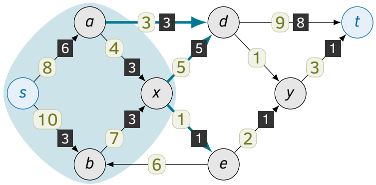
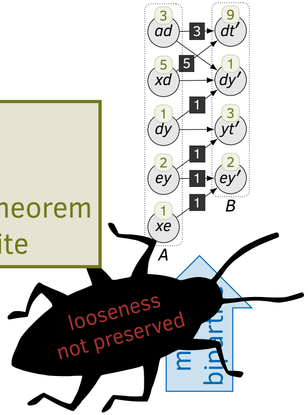
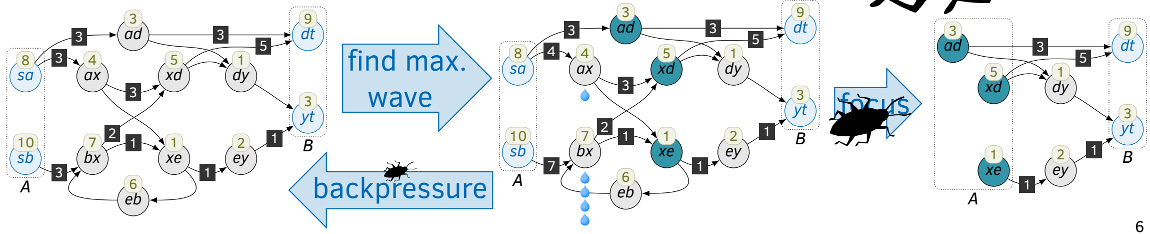
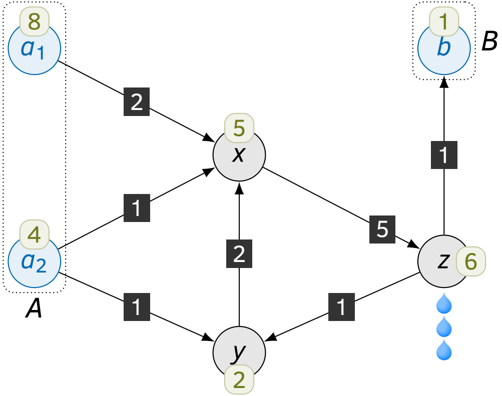


1. adapt proof to weakened induction invariant
2. new proof using finite MFMC theorem if total neighbour weight is finite





Backpressure fixpoint



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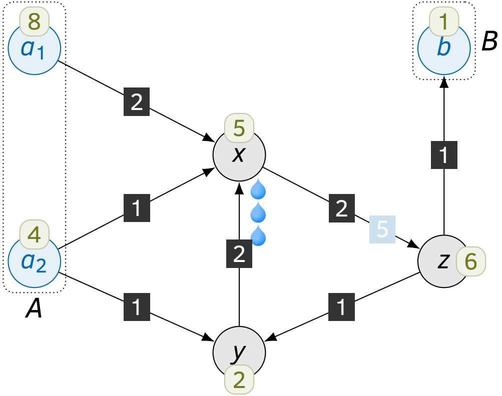

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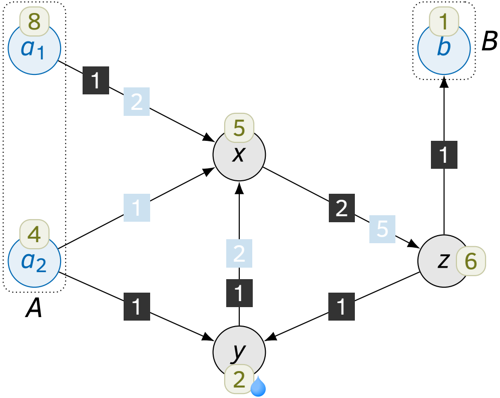

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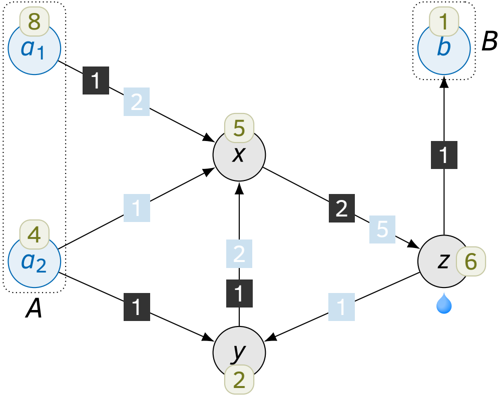

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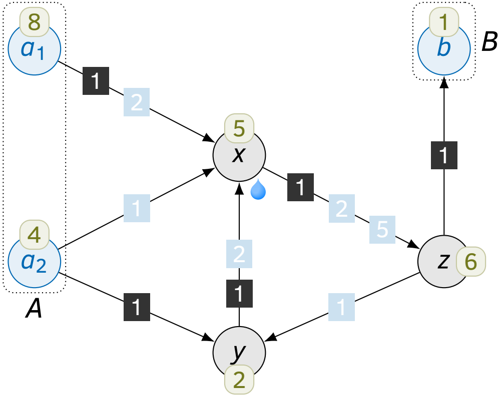

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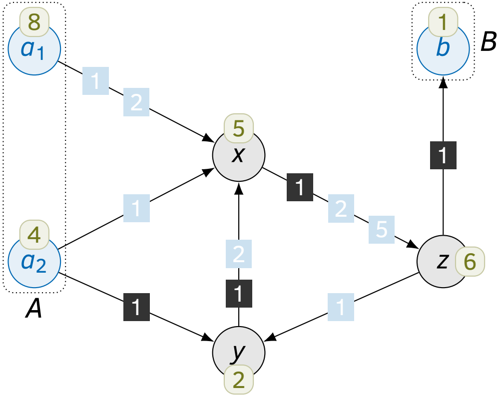

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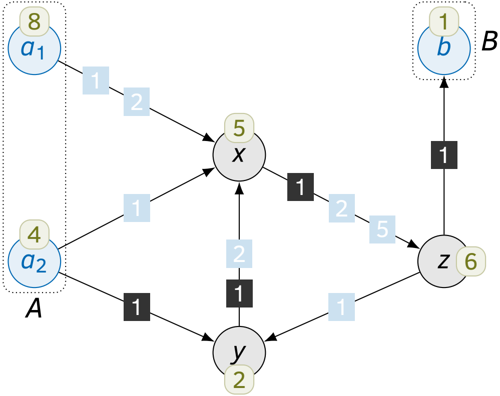

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Pick a leaking vertex 💧 if any
and reduce incoming flow.

$$f = \mathrm{fix}(bp_G)$$

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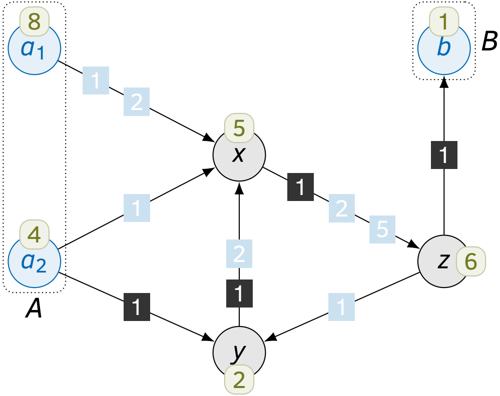

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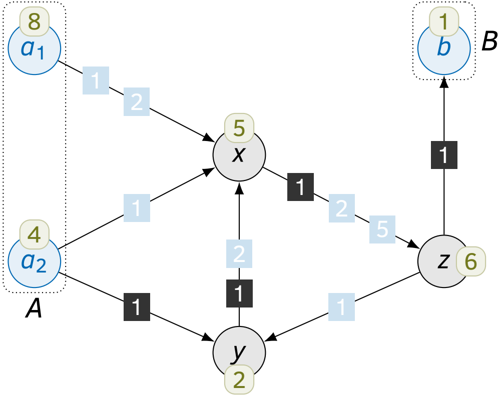

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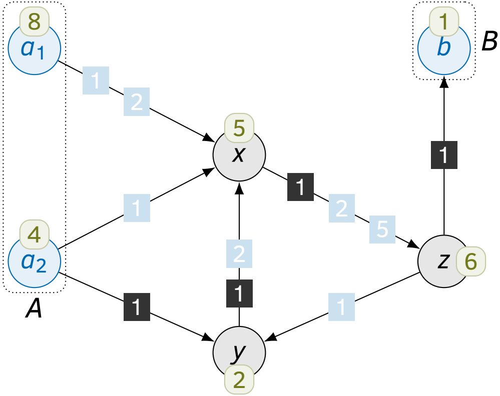

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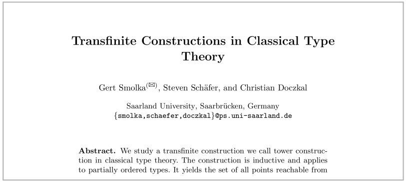

Transfinite Constructions in Classical Type Theory

Gert Smolka[✉], Steven Schäfer, and Christian Doczkal

Saarland University, Saarbrücken, Germany
{smolka,schaefer,doczkal}@ps.uni-saarland.de

Abstract. We study a transfinite construction we call tower construction in classical type theory. The construction is inductive and applies to partially ordered types. It yields the set of all points reachable from

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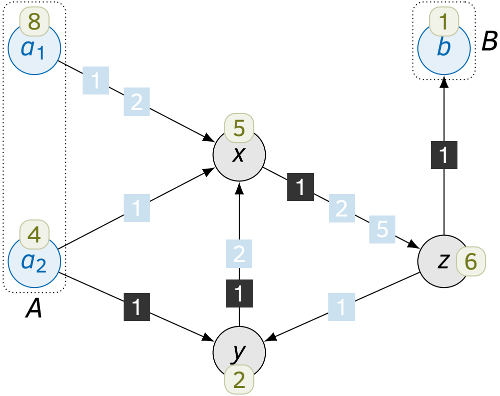

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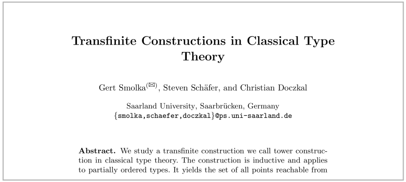
translate



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Summary: Avoid infinite sums!

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Available in the AFP

isa-afp.org/entries/MFMC_Countable.html

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	Line counts
Preliminaries	200
Networks & webs	2214
Transformations	1248
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Total	**10208**

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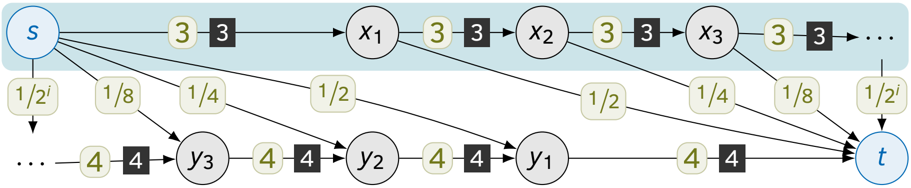

	value
cut	2
out-flow	3
in-flow	4